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Regular Almost-Periodic Functions and the Concept of Periodicity

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Abstract; The present paper investigates the concept of regular almost-periodic functions and their fundamental properties as a natural generalization of classical periodic functions. Special attention is paid to the notion of ε-periods and their role in describing quasi-periodic behavior on the real line. The study clarifies the mathematical essence of almost periodicity in the sense of Bohr and highlights the importance of relative density of ε-periods. Key structural properties of regular almost-periodic functions are discussed, emphasizing their stability under algebraic operations and translations. The relevance of these functions in the theory of differential equations and in the modeling of physical and technical systems is also briefly addressed. The results demonstrate that regular almost-periodic functions constitute an effective analytical tool for describing complex processes that cannot be adequately modeled using classical periodic functions.

***Keywords**: almost-periodic function, regularity, ε-period, Bohr almost periodicity, relative density*

1. INTRODUCTION

Classical periodic functions play a fundamental role in mathematical analysis and its applications, particularly in the study of oscillatory phenomena. However, many natural and technical processes exhibit behavior that is close to periodic but does not satisfy strict periodicity conditions. This limitation has motivated the development of broader concepts capable of capturing such phenomena more accurately.

One of the most influential generalizations of periodicity is the notion of almost-periodic functions, introduced by H. Bohr in the early twentieth century. Unlike strictly periodic functions, almost-periodic functions allow for approximate repetitions of values with arbitrary precision. This flexibility makes them especially suitable for modeling irregular oscillations encountered in differential equations, physics, and engineering.

A central idea in Bohr's theory is the concept of ε-periods, which replaces exact periodic shifts with approximate ones. The distribution of these ε-periods, particularly their relative density on the real line, plays a crucial role in determining the structural properties of almost-periodic functions.

The purpose of this paper is to study a specific class of almost-periodic functions, referred to as **regular almost-periodic functions**, and to analyze their main characteristics. The paper begins with essential definitions and preliminary concepts, then focuses on the generalization of periodicity through ε-periods and relatively dense sets. The structure of the paper is as follows: Section 2 introduces the basic definitions and terminology, while subsequent sections (not included here) investigate fundamental properties and applications of regular almost-periodic functions.

2. PRELIMINARIES AND BASIC DEFINITIONS

This section introduces the fundamental concepts and notation required throughout the paper. All functions considered are real-valued functions defined on the real line unless stated otherwise.

2.1. Generalization of the Period Concept

In classical analysis, a function ($f(x)$) is said to be periodic if there exists a nonzero real number (T) such that

$$f(x+T)=f(x) \quad \text{for all } x\in R.$$

While this definition is precise, it is often too restrictive for describing real-world phenomena.

To overcome this limitation, the concept of an **ε-period** is introduced. Let ($f:\mathbb{R} \to \mathbb{R}$) be a function and let ($\varepsilon > 0$). A real number (τ) is called an **ε-period** of (f) if

$$|f(x+\tau)-f(x)|<\varepsilon \quad \text{for all } x\in R.$$

Clearly, every classical period is an ε-period for any $\varepsilon>0$, but the converse is not necessarily true. The notion of ε-periods thus provides a natural extension of periodicity, allowing approximate repetitions of function values with arbitrary accuracy.

2.2. Relatively Dense Sets

The concept of relative density plays a key role in almost-periodic function theory.

A subset ($E \subset \mathbb{R}$) is called **relatively dense** if there exists a positive number ($l > 0$) such that every interval of length (l) on the real line contains at least one point of (E).

For example, the set

$$E = \{np: n = 0, \pm 1, \pm 2, \ldots \},$$

where $p > 0$, is relatively dense. In contrast, the set

$$E = \{\pm n^2: n = 1,2,3, \ldots \}$$

is not relatively dense, since the gaps between consecutive elements grow without bound.

In the theory of almost-periodic functions, relative density ensures that ε-periods occur frequently enough across the real line. This property guarantees uniform recurrence and lies at the heart of Bohr's definition of almost periodicity.

3. DEFINITION OF REGULAR ALMOST-PERIODIC FUNCTIONS

The central object of this study is the class of regular almost-periodic functions, which refines Bohr's notion of almost periodicity by emphasizing uniform recurrence and continuity.

Definition 3.1. (ε-Almost Period)

Let $f: \mathbb{R} \to \mathbb{R}$ be a function and let $\varepsilon > 0$. A real number τ is called an **ε-almost period** of f if

$$| f(x + \tau) - f(x) | < \varepsilon \text{ for all } x \in \mathbb{R}.$$

Definition 3.2. (Regular Almost-Periodic Function)

A continuous function $f: \mathbb{R} \to \mathbb{R}$ is called a **regular almost-periodic function** if, for every $\varepsilon > 0$, the set of ε-almost periods of f is relatively dense in ℝ.

Equivalently, for each $\varepsilon > 0$ there exists a number $l = l(\varepsilon) > 0$ such that every interval of length l contains at least one real number τ satisfying

$$| f(x + \tau) - f(x) | < \varepsilon \text{ for all } x \in \mathbb{R}.$$

The regularity condition ensures that approximate repetitions of the function values occur uniformly along the real axis. Intuitively, although the function may not repeat itself exactly, it does so with arbitrary precision and without large gaps between repetitions.

This definition coincides with Bohr's classical concept of almost-periodic functions on ℝ, emphasizing uniform convergence and relative density of ε-periods. Hence, regular almost-periodic functions may be viewed as a natural and analytically convenient generalization of periodic functions.

4. FUNDAMENTAL PROPERTIES OF REGULAR ALMOST-PERIODIC FUNCTIONS

In this section, we establish basic closure and stability properties of regular almost-periodic functions.

Proposition 4.1. (Scalar Multiplication Invariance)

If $f(x)$ is a regular almost-periodic function and $\alpha \in \mathbb{R}$, then the function $\alpha f(x)$ is also regular almost-periodic.

Proof.
Let $\varepsilon > 0$. Since f is regular almost-periodic, there exists an ε-almost period τ such that

$$| f(x + \tau) - f(x) | < \frac{\varepsilon}{| \alpha |} \text{ for all } x \in \mathbb{R}.$$

Then

$$| \alpha f(x + \tau) - \alpha f(x) | = | \alpha | \; | f(x + \tau) - f(x) | < \varepsilon,$$

which proves the claim.

Proposition 4.2. (Translation Invariance)

If $f(x)$ is a regular almost-periodic function and $c \in \mathbb{R}$, then $f(x + c)$ is also regular almost-periodic.

Proof.
For any ε-almost period τ of f, we have

$$| f(x+c+\tau) - f(x+c) | < \varepsilon$$

for all $x \in \mathbb{R}$. Hence, the same ε-almost periods apply to the translated function.

Proposition 4.3. (Closure under Absolute Value)

If $f(x)$is a regular almost-periodic function, then $| f(x) |$is also regular almost-periodic.

Proof.
Using the inequality

$$|| f(x+\tau) | - | f(x) || \leq | f(x+\tau) - f(x) |,$$

every ε-almost period of $f$is also an ε-almost period of $| f |$.

Proposition 4.4. (Closure under Reciprocal)

Let $f(x)$be a regular almost-periodic function satisfying

$$\inf_{x \in \mathbb{R}} | f(x) | = \gamma > 0.$$

Then the function $1/f(x)$is regular almost-periodic.

Proof.
For an ε-almost period $\tau$of f,

$$| \frac{1}{f(x+\tau)} - \frac{1}{f(x)} | = \frac{| f(x+\tau) - f(x) |}{| f(x+\tau) f(x) |} \leq \frac{1}{\gamma^2} | f(x+\tau) - f(x) |.$$

Thus, ε-almost periods of $f$yield ε-almost periods of $1/f$.

Proposition 4.5. (Closure under Composition)

Let $f(x)$be regular almost-periodic and let Fbe uniformly continuous on the range of f. Then the composite function $F(f(x))$is regular almost-periodic.

Proof.
Uniform continuity of Fimplies that small variations in $f(x)$lead to small variations in $F(f(x))$. Hence, ε-almost periods of $f$induce ε-almost periods of $F \circ f$.

Proposition 4.6. (Boundedness)

Every regular almost-periodic function is bounded on ℝ.

Proof.
Let $\varepsilon = 1$and let $l = l(1)$be the corresponding length ensuring ε-almost periods. Since $f$is continuous on a compact interval of length l, it attains a maximum M. Using ε-almost periods, the bound extends to all $x \in \mathbb{R}$.

5. CONSEQUENCES AND DERIVED RESULTS

The fundamental properties established above lead to several important corollaries.

Corollary 5.1. (Square of an Almost-Periodic Function)

If $f(x)$is regular almost-periodic, then $f^2(x)$is also regular almost-periodic.

Proof.
Using boundedness and the identity

$$| f^2(x+\tau) - f^2(x) | = | f(x+\tau) - f(x) | \quad | f(x+\tau) + f(x) |,$$

the claim follows directly.

Corollary 5.2. (Stability under Algebraic Operations)

The class of regular almost-periodic functions is closed under addition, subtraction, and multiplication.

Corollary 5.3. (ε-Period Transformation)

ε-almost periods of a regular almost-periodic function remain ε-almost periods under scalar multiplication, translation, and uniformly continuous transformations.

These results confirm that regular almost-periodic functions form a robust and analytically stable class, well-suited for applications in differential equations and mathematical modeling.

6. UNIFORM CONTINUITY OF REGULAR ALMOST-PERIODIC FUNCTIONS

One of the fundamental analytical properties of regular almost-periodic functions is uniform continuity on the entire real line. This property is essential for both theoretical investigations and practical applications.

Theorem 6.1. (Uniform Continuity)

Every regular almost-periodic function $f: \mathbb{R} \to \mathbb{R}$is uniformly continuous on $\mathbb{R}$.

Proof.
Let $\varepsilon > 0$be arbitrary. Since $f$is regular almost-periodic, there exists $l = l(\varepsilon/3) > 0$such that every interval of length $l$contains an ε/3-almost period τsatisfying

$$| f(x+\tau) - f(x) | < \varepsilon/3 \text{for all } x \in \mathbb{R}.$$

Because $f$is continuous on the compact interval $[0, l]$, it is uniformly continuous there. Hence, there exists $\delta > 0$such that

$$| x_1 - x_2 | < \delta \Rightarrow | f(x_1) - f(x_2) | < \varepsilon/3$$

for all $x_1, x_2 \in [0, l]$.

Now let $x, y \in \mathbb{R}$with $| x - y | < \delta$. Choose an ε/3-almost period τsuch that both $x + \tau$and $y + \tau$lie in $[0, l]$. Then

$$
\begin{aligned}
|f(x)-f(y)| &\le |f(x)-f(x+\tau)| + |f(x+\tau)-f(y+\tau)| + |f(y+\tau)-f(y)| \\
&< \varepsilon/3+\varepsilon/3+\varepsilon/3=\varepsilon.
\end{aligned}
$$

Thus, f is uniformly continuous on $\mathbb{R}$.

Uniform continuity guarantees stability of regular almost-periodic functions under limits and ensures well-posedness in analytical and numerical applications, particularly in differential equations.

7. APPLICATIONS AND CONNECTIONS

Regular almost-periodic functions play a significant role in various branches of mathematics and applied sciences.

7.1. Differential Equations

In the theory of ordinary and partial differential equations, almost-periodic coefficients naturally arise in models with recurring but non-strictly periodic behavior. Regular almost-periodic functions provide a suitable framework for studying existence, uniqueness, and stability of solutions, especially in non-autonomous systems.

7.2. Applications in Physics

Many physical systems exhibit oscillations that are not exactly periodic, such as wave propagation in inhomogeneous media or forced vibrations with varying frequencies. Regular almost-periodic functions allow accurate modeling of such phenomena by capturing approximate repetitions without requiring strict periodicity.

7.3. Engineering and Technical Systems

In engineering, regular almost-periodic functions are applied in signal processing, control theory, and mechanical system modeling. They offer advantages over strictly periodic models by accommodating real-world irregularities while preserving analytical tractability.

7.4. Advantages over Classical Periodic Models

Compared to classical periodic functions, regular almost-periodic functions provide greater flexibility and realism. The ε-period framework ensures robustness under perturbations and makes these functions particularly suitable for complex systems.

8. DISCUSSION

The results obtained in this paper place regular almost-periodic functions within the broader context of almost-periodic function theory. The definition adopted here aligns with Bohr's classical approach, emphasizing uniform convergence and relative density of ε-periods.

Compared to alternative formulations, such as Besicovitch or Stepanov almost periodicity, regular almost-periodicity ensures stronger regularity properties, including boundedness and uniform continuity. The connections with the works of Bohr, Corduneanu, and Levitan highlight the foundational nature of this concept and its relevance to differential equations.

Possible extensions of the present study include:

- Almost-periodic functions in Banach and Hilbert spaces
- Almost-periodic solutions of nonlinear differential equations
- Generalizations to time scales and stochastic frameworks

These directions suggest that regular almost-periodic functions remain an active and promising area of research.

9. CONCLUSION

This paper has presented a systematic study of regular almost-periodic functions and their fundamental properties. By generalizing the classical notion of periodicity through ε-periods and relative density, a robust and flexible analytical framework has been established.

The main contributions include:

- A precise definition of regular almost-periodic functions
- Proof of closure, boundedness, and uniform continuity properties
- Discussion of applications in differential equations and applied sciences

The ε-period approach proves to be a powerful tool for modeling and analysis, bridging the gap between strict periodicity and real-world quasi-periodic behavior. Future research may further expand these results to abstract spaces and complex dynamical systems.

REFERENCES

Bohr, H. (1947). *Almost Periodic Functions*. New York: Chelsea Publishing Company.

Bohr, H. (1925). *Zur Theorie der fastperiodischen Funktionen. I. Eine Verallgemeinerung der Theorie der Fourierreihen*. *Acta Mathematica, 45*(1), 29–127.

Bohr, H. (1925). *Zur Theorie der Fastperiodischen Funktionen. II. Zusammenhang der fastperiodischen Funktionen mit Funktionen von unendlich vielen Variabeln; gleichmässige Approximation durch trigonometrische Summen*. *Acta Mathematica, 46*(1–2), 101–214.

Bohr, H. (1926). *Zur Theorie der fastperiodischen Funktionen. III. Dirichletentwicklung analytischer Funktionen*. *Acta Mathematica, 47*(3), 237–281.

Besicovitch, A. S. (1932). *Almost Periodic Functions*. Cambridge: Cambridge University Press.

Corduneanu, C. (1968). *Almost Periodic Functions* (Interscience Tracts in Pure and Applied Mathematics, No. 22). New York: Interscience Publishers.

Levitan, B. M., & Zhikov, V. V. (1982). *Almost Periodic Functions and Differential Equations*. Cambridge: Cambridge University Press.

Fink, A. M. (1974). *Almost Periodic Differential Equations* (Lecture Notes in Mathematics, Vol. 377). Berlin–Heidelberg: Springer.

Bochner, S. (1962). *A New Approach to Almost Periodicity*. *Proceedings of the National Academy of Sciences of the United States of America, 48*(12), 2039–2043. https://doi.org/10.1073/pnas.48.12.2039

Amerio, L., & Prouse, G. (1971). *Almost-Periodic Functions and Functional Equations*. New York: Van Nostrand Reinhold.

N'Guérékata, G. M. (2001). *Almost Automorphic and Almost Periodic Functions in Abstract Spaces*. New York: Springer.

Diagana, T. (2013). *Almost Automorphic Type and Almost Periodic Type Functions in Abstract Spaces*. Cham: Springer. https://doi.org/10.1007/978-3-319-00849-3

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