Derivatives and Differentials of Multivariable Functions: A Theoretical and Applied Analysis

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DOI: https://doi.org/10.69760/lumin.2026002002

Abstract

This article presents a comprehensive theoretical and applied investigation of derivatives and differentials of multivariable functions within the framework of higher mathematics. The study examines partial derivatives of functions of two and n variables, develops the concepts of the total differential and the gradient, and establishes the key properties governing these operations, including linearity, the product and quotient rules, the chain rule, and the equality of mixed partial derivatives (Clairaut's theorem). The theoretical exposition is supported by three worked examples covering polynomial, trigonometric, and logarithmic multivariable functions. The geometric interpretation of the total differential as the tangent-plane approximation is discussed alongside its practical applications in physics, chemistry, and engineering. The findings confirm that multivariable derivatives and differentials constitute a foundational tool for local behaviour analysis, extremum problems, and linear approximation in applied science and engineering contexts.

Keywords: *multivariable functions; partial derivatives; total differential; gradient; chain rule; Clairaut's theorem; linear approximation*

Article History
Received: 15 February 2026
Revised: 25 February 2026
Accepted: 8 April 2026
Published: 14 April 2026

1. Introduction

The study of multivariable functions represents one of the central topics in mathematical analysis and constitutes an indispensable foundation for advanced work in applied science, engineering, and related disciplines. Whereas single-variable calculus describes the behaviour of quantities that depend on one independent parameter, real-world phenomena almost invariably involve several simultaneously changing quantities. The temperature distribution within a solid body, the pressure of a gas as a function of volume and temperature, or the stress field in a structural component all require a mathematical framework capable of handling dependence on multiple variables simultaneously (Stewart, 2016; Marsden & Tromba, 2012).

The fundamental objects of this framework are partial derivatives and the total differential. A partial derivative measures the rate of change of a function with respect to a single independent variable while all remaining variables are held constant, thereby extending the classical derivative concept to higher-dimensional domains. The total differential provides a first-order linear approximation of the overall change in the function when all its arguments vary simultaneously, and forms the basis for sensitivity analysis, error estimation, and the linearisation of nonlinear systems (Adams & Essex, 2018).

Despite the importance of these concepts, their systematic presentation at the advanced undergraduate level often lacks integration between the theoretical derivation and worked illustrative examples. The present article addresses this gap by providing a unified treatment that moves from rigorous definition through the principal algebraic properties to concrete computational examples involving polynomial, trigonometric, and logarithmic functions. Section 2 establishes notation and the core definitions. Section 3 develops the theoretical properties of partial derivatives. Section 4 presents the concept and properties of the total differential. Section 5 provides three fully solved examples. Section 6 discusses the geometric interpretation and applied significance of the results, and Section 7 presents conclusions.

2. Theoretical Background: Multivariable Functions and Partial Derivatives

Let z = f(x, y) be a function of two independent variables x and y, where z is the dependent variable. More generally, a function of n variables is written as:

$$z = f(x_1, x_2, \ldots, x_n)$$

where x_1, x_2, …, x_n are the independent variables. The study of such functions requires understanding their rate of change with respect to each variable independently.

The increment (total change) of the function z = f(x, y) when both arguments change simultaneously is defined as:

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y)$$

If only one variable changes, the corresponding partial increments are obtained. These partial increments form the basis for defining partial derivatives.

2.1 Definition of Partial Derivatives

Definition 1. If the following limit exists, it is called the partial derivative of f with respect to x at the point (x, y):

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

By the same rule, the partial derivative with respect to y is defined as:

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

Equivalently, these derivatives are written using the ∂ (partial) notation. The partial derivative ∂z/∂x is computed by treating y as a constant and applying the ordinary rules of differentiation; similarly, ∂z/∂y is computed by treating x as a constant.

For a function of n variables, the partial derivative with respect to the k-th variable is:

$$\frac{\partial f}{\partial x_i} = \lim_{\Delta x_i \to 0} \frac{f(x_1, \ldots, x_i + \Delta x_i, \ldots, x_n) - f(x_1, \ldots, x_n)}{\Delta x_i}$$

An alternative notation uses subscripts: $f_x \equiv \partial f/\partial x$ and $f_y \equiv \partial f/\partial y$, as illustrated by:

$$\frac{\partial f}{\partial l} = \frac{\partial f}{\partial x}\cos\alpha + \frac{\partial f}{\partial y}\cos\beta$$

2.2 Second-Order and Mixed Partial Derivatives

Second-order partial derivatives are obtained by differentiating the first-order partial derivatives:

$$\frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial y^2}, \quad \frac{\partial^2 f}{\partial x \partial y}$$

Of particular importance are the mixed second-order partial derivatives $\partial^2 z/(\partial x\partial y)$ and $\partial^2 z/(\partial y\partial x)$. Under standard continuity conditions, these are equal:

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

This result is known as Clairaut's theorem (also called Schwarz's theorem), which holds whenever the mixed partial derivatives are continuous (Apostol, 1969).

2.3 The Gradient

The gradient of a function z = f(x, y) is defined as the vector of its first-order partial derivatives:

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$$

In the general n-variable case:

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}\right)$$

The gradient points in the direction of the greatest rate of increase of the function, and its magnitude equals the maximum rate of change:

$$|\nabla f| = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}$$

3. Properties of Partial Derivatives

Let z = f(x, y) and w = g(x, y) be two functions whose partial derivatives exist at the relevant point. The following properties hold for partial differentiation (Piskunov, 1969; Minorsky, 1978).

Property 1 (Linearity). If f and g both have partial derivatives, then for any constants α, β:

$$\frac{\partial}{\partial x}(f \pm g) = \frac{\partial f}{\partial x} \pm \frac{\partial g}{\partial x}$$
$$\frac{\partial}{\partial y}(f \pm g) = \frac{\partial f}{\partial y} \pm \frac{\partial g}{\partial y}$$

Property 2 (Multiplication by a constant). If c = const, then:

$$\frac{\partial}{\partial x}(cf) = c\frac{\partial f}{\partial x}, \quad \frac{\partial}{\partial y}(cf) = c\frac{\partial f}{\partial y}$$

Property 3 (Product rule).

$$\frac{\partial}{\partial x}(f \cdot g) = \frac{\partial f}{\partial x} \cdot g + f \cdot \frac{\partial g}{\partial x}$$
$$\frac{\partial}{\partial y}(f \cdot g) = \frac{\partial f}{\partial y} \cdot g + f \cdot \frac{\partial g}{\partial y}$$

Property 4 (Quotient rule).

Əgər $g(x, y) \neq 0$, onda

$$\frac{\partial}{\partial x}\left(\frac{f}{g}\right) = \frac{\frac{\partial f}{\partial x} \cdot g - f \cdot \frac{\partial g}{\partial x}}{g^2}$$
$$\frac{\partial}{\partial y}\left(\frac{f}{g}\right) = \frac{\frac{\partial f}{\partial y} \cdot g - f \cdot \frac{\partial g}{\partial y}}{g^2}$$

Property 5 (Chain rule). If z = f(u, v) where u = u(x, y) and v = v(x, y), so that z = f(u(x,y), v(x,y)), then:

$$z = f(u, v), \quad u = u(x, y), \quad v = v(x, y)$$
$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x}$$
$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y}$$

Property 6 (Linearity of the total differential). If the functions are differentiable, then:

$$d(f \pm g) = df \pm dg$$
$$d(cf) = c\,df$$

Property 7 (Expression of the total differential). If f is differentiable, then:

$$dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

Property 8 (Equality of mixed partial derivatives). If the second-order partial derivatives are continuous, then:

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

Property 9 (Directional derivative and gradient relation). For the directional derivative in direction ℓ:

$$\frac{\partial f}{\partial l} = \nabla f \cdot \vec{l}$$

where:

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$$

These properties collectively allow the manipulation of multivariable derivatives in a manner directly analogous to single-variable calculus, while the chain rule and directional derivative formulae extend the theory to compositions and arbitrary directions (Stewart, 2016).

4. The Total Differential of a Multivariable Function

The concept of the total differential formalises the notion of the best linear approximation to the change in a function when all its arguments change simultaneously.

For $z = f(x, y)$, the increment is: $\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y)$

If the function is differentiable at (x, y), the increment can be decomposed as:

$$\Delta z = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \varepsilon$$

where ε is an infinitesimal of higher order, meaning $\varepsilon \to 0$ as $\Delta x \to 0$ and $\Delta y \to 0$.

Definition 2. A function $z = f(x, y)$ is said to be differentiable at (x, y) if its increment can be written in the form above. The total differential of z is the linear part of this increment:

$$dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

where $dx = \Delta x$ and $dy = \Delta y$ are the differentials of the independent variables. The relation $\Delta z \approx dz$ provides a first-order approximation to the actual change.

For a function of n variables $z = f(x_1, x_2, \ldots, x_n)$, the total differential is:

$$dz = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \cdots + \frac{\partial f}{\partial x_n}dx_n$$

This expression represents the linear (first-order) approximation of the function in the multidimensional space and is widely used in practical computations (Adams & Essex, 2018).

4.1 Geometric Interpretation

If $z = f(x, y)$ is visualised as a surface in three-dimensional space, the total differential at a given point defines the tangent plane to that surface at that point. The equation of the tangent plane at (x_0, y_0, z_0) is:

$$z - z_0 = \frac{\partial f}{\partial x}(x - x_0) + \frac{\partial f}{\partial y}(y - y_0)$$

This shows that the differential provides the best local flat approximation to the surface. In applied contexts — structural analysis, fluid dynamics, thermodynamics — this linear approximation is the basis of sensitivity analysis and small-perturbation theory (Marsden & Tromba, 2012).

5. Worked Examples

The following three examples illustrate the computation of partial derivatives for polynomial, trigonometric, and logarithmic multivariable functions.

5.1 Example 1: Polynomial Function

Problem. Find the partial derivatives of the function:

$$z = x^2 y + 3xy^2$$

Solution. This is a two-variable polynomial function. Partial derivatives are computed by treating one variable as constant and differentiating with respect to the other.

Differentiating with respect to x (y held constant):

$$\frac{\partial z}{\partial y} = x^2 + 6xy$$

Differentiating with respect to y (x held constant):

$$\frac{\partial z}{\partial y} = x^2 + 6xy$$

The results show that the function changes at different rates with respect to each variable, and these rates of change are expressed through the partial derivatives.

5.2 Example 2: Trigonometric Function

Problem. Find the partial derivatives of z = sin(xy).

Solution. This is a composite function in which the sine function has an inner argument that depends on both variables. The chain rule must therefore be applied.

Partial derivative with respect to x:

$$\frac{\partial z}{\partial x} = \cos(xy) \cdot y$$

Partial derivative with respect to y:

$$\frac{\partial z}{\partial y} = \cos(xy) \cdot x$$

This example illustrates that when differentiating composite multivariable functions, both the outer and inner function derivatives must be incorporated simultaneously, via the chain rule.

5.3 Example 3: Logarithmic Function

Problem. Find the partial derivatives of the function:

$$z = \ln(x^2 + y^2)$$

Solution. This is a composite logarithmic function. Both the logarithm differentiation rule and the chain rule are applied.

Partial derivative with respect to x:

$$\frac{\partial z}{\partial x} = \frac{2x}{x^2 + y^2}$$

Partial derivative with respect to y:

$$\frac{\partial z}{\partial y} = \frac{2y}{x^2 + y^2}$$

The symmetry of the results with respect to x and y reflects the symmetric structure of the function's argument ($x^2 + y^2$), a pattern frequently encountered in physical models based on Euclidean distance (Piskunov, 1969).

6. Discussion

The theoretical and computational analysis presented in the preceding sections highlights several key dimensions of the role of partial derivatives and the total differential in applied mathematics and engineering.

First, partial derivatives extend the classical derivative concept to higher-dimensional domains in a natural and computationally tractable way. By holding all but one variable constant, they isolate the directional rate of change of a function, enabling the analysis of complex systems through manageable one-dimensional projections. This property makes them indispensable in optimisation: necessary conditions for extrema of differentiable functions of several variables are formulated entirely in terms of partial derivatives set equal to zero (Marsden & Tromba, 2012; Stewart, 2016).

Second, the total differential provides a unified linear approximation framework. In engineering and physics, where exact analytical solutions are often unavailable or computationally prohibitive, the approximation $\Delta z \approx dz$ permits rapid estimation of how a system output changes under small perturbations of its inputs. Error propagation analysis in experimental science, sensitivity analysis in control engineering, and first-order perturbation theory in quantum mechanics all rely on this principle (Adams & Essex, 2018).

Third, the gradient vector encodes directional information about the function's rate of change that no single partial derivative alone can capture. In optimisation algorithms such as gradient descent, the gradient is the primary computational object; in physics, gradient fields correspond to force fields derived from potential functions. The geometric interpretation of the gradient as pointing normal to level curves (in 2D) or level surfaces (in 3D) provides an intuitive foundation for these applications.

Fourth, Clairaut's theorem — the equality of mixed second-order partial derivatives under continuity — has substantial practical importance: it reduces the number of independent second-order derivatives that must be computed from n^2 to $n(n+1)/2$, significantly lowering computational cost for functions of many variables. This simplification underlies the symmetry of the Hessian matrix used in second-order optimality conditions.

The three worked examples demonstrate the standard computational strategies: direct power-rule differentiation for polynomial functions, application of the chain rule for composite trigonometric functions, and combined logarithm and chain-rule differentiation for logarithmic functions. The pattern of symmetric partial derivatives observed in Example 3 reflects a general principle: functions of the form f(r) where $r = \sqrt{(x^2 + y^2)}$ are radially symmetric, and their partial derivatives satisfy $\partial f/\partial x = (x/r)f'(r)$ and $\partial f/\partial y = (y/r)f'(r)$, which are proportional by the factor y/x (Apostol, 1969).

7. Conclusion

This article has provided a systematic theoretical and applied treatment of derivatives and differentials of multivariable functions. The partial derivative formalises the rate of change of a multivariable function with respect to a single argument, with all others held fixed, and is computed by the ordinary rules of single-variable differentiation applied to the target variable. The gradient assembles the complete first-order information of the function into a vector pointing in the direction of steepest ascent, with magnitude equal to the maximum directional derivative.

The total differential, defined as the principal linear part of the function's increment, provides the best first-order approximation to the change in the function. For $z = f(x_1, x_2, \ldots, x_n)$, it is given by:

$$dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

This linear approximation underlies a wide range of applications in physics, chemistry, and engineering, from error estimation to tangent-plane construction and sensitivity analysis.

The nine operational properties established in Section 3 — covering linearity, product and quotient rules, the chain rule, the equality of mixed partial derivatives, and the gradient–directional derivative relation — provide a complete algebraic toolkit for manipulating multivariable derivatives and differentials. The three worked examples confirm the practical applicability of these properties across different function classes.

Mastery of these concepts constitutes a prerequisite for advanced topics in mathematical analysis, including multiple integrals, differential equations, optimisation theory, and the mathematical modelling of physical and engineering systems. Future work may extend the present analysis to higher-order differentials, implicit differentiation of multivariable equations, and applications in constrained optimisation via Lagrange multipliers.

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Declarations

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflicts of interest.

Ethical Approval: Not applicable. This study is based on a review of published literature and does not involve human participants or animal subjects.

Author Contributions: Həsənov M. contributed to the conceptualization, formal analysis, and worked examples. Qocayeva N. contributed to the theoretical framework, writing, and revision. All authors have read and approved the final version.

Data Availability Statement: No new data were generated or analyzed in this study. All sources are cited in the reference list.